

Identification of Nonlinear Hammerstein-Wiener Fractional Order System with Nonzero Initial Conditions by Block Impulse Function

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ARTICLE INFORMATION	ABSTRACT
Article history: Published: October 2026 Keywords: Nonlinear Fractional Order Systems System Identification Fractional Integral Operational Matrices Recursive Least Squares Initial State Estimation	In this paper, we present an identification method for nonlinear fractional order systems with nonzero initial conditions. The nonlinear functions are approximated by block impulse functions and by Riemannian-Liouville fractional integral matrix, the nonlinear continuous fractional order system is transformed into an algebraic system. Then, the initial conditions of the system states are treated as unknown parameters in known nonlinear form. Applying some techniques, we obtain regression equations to estimate the linear parameters and initial conditions of the system in the case where the derivative coefficients of the nonlinear fractional order system are known. For initial conditions and parameter estimation, a recursive least squares method was applied. The proposed method has been validated through several simulations.

1. Introduction

Fractional order systems using fractional order derivatives have been widely used in practice because they represent many engineering phenomena accurately [1-3]. The applications of viscoelastic phenomena in mechanics [4], ferromagnetic hysteresis [5], anomalous thermal conduction in porous media [6], lithium battery charging and discharging characteristics [7], and ultracapacitors electrical characteristics [8], etc. are increasing. In mechanics, fractional calculus is essentially used to model the constitutive law of viscous media. Since [9,10] first proposed the application of fractional calculus in viscoelastic media, fractional calculus has been applied to modeling of viscoelastic dampers for structural vibration energy dissipation [11,12] and modeling of rubber bearings in vibration isolated structures [13]. Since fractional order derivatives are not explicitly given except for a few phenomena of particular physical meaning, systems modeling using fractional order derivatives generally relies on identification methods based on input and output data in many cases. Generally, fractional order systems identification methods are developed in two directions. One is the identification method in the frequency domain [14,15] and the other is the identification method in the time domain [16-23]. Recently, identification methods of fractional order systems have been developed more in the time domain than in the frequency domain. This is because the identification method in the time domain is more analytical, but it can estimate parameters that cannot be estimated in the frequency domain, including the initial states of the system and the time delay. Initially, the study of fractional order systems identification was studied in continuous systems, but since we realize that this method has many shortcomings, there is an increasing interest in the identification of the transformation into an algebraic system. The transformation into an algebraic system is mainly based on block impulse functions(BPFs) [16,17], and many methods have been developed to increase the identification accuracy by hybridizing with special polynomials such as Haar wavelets [18], Legendre polynomials [19], and Bernoulli polynomials [20].

In the past, many achievements have been made in the identification methods for linear fractional order systems.

Although many studies have been reported on the identification of linear fractional order systems, the main results can be summarized as follows. In [21], a method for estimating the basic parameters and input delay of linear fractional order systems based on block impulse functions (BPFs) is proposed, while in [22] a method for estimating the basic parameters and initial state of linear subsystem is proposed. Based on this, in [23], a method for estimating the basic parameters, input delay, and both initial states and fractional order of a linear fractional order systems is proposed. In addition, many methods have been proposed, but in many cases, the studies that improve the accuracy of system identification are mostly.

However, there are many phenomena modeled as nonlinear fractional order systems in practice. In [24], the vibration transfer characteristics of a single-degree-of-freedom oscillator with nonlinear fractional order damping are analyzed, and in [25] the vibration analysis of a multi-degree-of-freedom nonlinear fractional order system with fractional order derivative is carried out. Also, in [26], a nonlinear fractional hysteresis loop connected in parallel with a fractional capacitor and a memristor is analyzed. In practical situations, there are more cases of modeling a nonlinear fractional order systems than a phenomenon modeled as a linear fractional order system. Therefore, there is a real need to study nonlinear fractional order systems identification methods. In general, the development of identification methods for nonlinear fractional order systems has been considered a difficult problem, and there is not much literature on these problems in the past, and there are still few problems to be solved.

The identification method of the nonlinear fractional Hammerstein model proposed in [27] considers a linear fractional order systems as a subsystem and the identification method in which the input signal is obtained via a nonlinear subsystem. Thus, this method is essentially a linear fractional order systems identification method. In [28], a fractional algebraic identifiability is introduced to represent the system parameters as a function of the inputs and outputs of the system for nonlinear fractional order systems

identification, based on which a fractional proportional-integral estimator with global Mittag-Leffler boundedness is constructed and parameter estimation is performed.

This method has great advantages for identifying time-varying nonlinear fractional order systems. However, the introduction of parameters satisfying non-contractive fractional algebraic observability has the disadvantage that the method is not deterministic explicitly and does not estimate the initial state. The biggest difficulty of this method is the need to construct a fractional PID estimator that satisfies the global Mittag-Leffler boundedness for parameter estimation of nonlinear fractional order systems, which means that additional theoretical studies must be carried out with the identification objective for nonlinear fractional order systems identification. In [28], a method for identifying the main parameters, input delay and fractional order of nonlinear fractional order systems is presented. In this way, we propose a method to estimate the input delay of nonlinear fractional order systems, unlike in the past. Although Legendre wavelet bases are used to improve the identification accuracy, the essential advantage is to improve the identification accuracy by constructing the time delay matrix as a continuous delay matrix with continuous functions as elements. However, this method does not estimate the initial state of the system. This is because this method uses the fractional order derivative operator matrix for nonlinear fractional order systems with Riemann-Liouville derivatives. Of course, the input time delay has a large effect on the output of the nonlinear fractional order systems, but the initial state has no less effect.

In this paper, we propose a method to estimate the basic parameters and initial states of a nonlinear Hammerstein-Wiener fractional order system. The main idea of this paper is to consider a nonlinear Hammerstein-Wiener fractional order systems, expressed as Caputo derivative, to estimate the initial state of a nonlinear fractional order systems, and to propose an identification model to estimate the initial state of a nonlinear fractional order systems based on the Riemannian Liouville fractional integral operational matrix. We used the fractional integral operator matrix with the block impulse function as the basis. Of course, we may use a hybrid basis with several special polynomials to improve the identification accuracy, but this does not have any essential significance. The paper is organized as follows. Section 2 gives the mathematical knowledge related to the paper, such as fractional calculus and fractional integral operational matrices, and Section 2 gives the main results. Section 3 demonstrates the effectiveness of the proposed method with a scientific example. Section 4 concludes the paper.

2. Preliminary Knowledge

2.1 Fractional Calculus

The fractional calculus is a generalization of the integral calculus of the integer order. The definition of the fractional order integral of Riemann-Liouville (R-L) is as follows [1].

$$\left({}^{RL}I_t^\alpha f\right)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, \tau > t_0 \\ f(t), & \alpha = 0 \end{cases} \quad (1)$$

where $\Gamma(\cdot)$ is a gamma function and $n-1 < \alpha \leq n, n \in \mathbf{N}$. For simplicity, ${}^{RL}I_t^\alpha$ is denoted I^α when $t_0 = 0$ is used.

The Riemann-Liouville fractional integral satisfies the following semigroup property.

$$\left(I^{\alpha+\beta} f\right)(t) = \left(I^\alpha \left(I^\beta f\right)\right)(t), \quad \alpha > 0, \beta > 0 \quad (2)$$

Also, the Riemann-Liouville $k, (k \in \mathbf{N})$ order integral of constant number is as follows

$$I^k(C) = \frac{t^k}{k!} C \quad (3)$$

In this paper, we use the Caputo fractional derivative defined as follows.

$$\left({}^CD_t^\alpha f\right)(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & \alpha > 0, \tau > t_0 \\ \frac{d^n f(t)}{dt^n}, & \alpha = n \end{cases} \quad (4)$$

We denote the Caputo derivative ${}^CD_t^\alpha$ in the case of $t_0 = 0$ as D^α for simplicity.

We use the following relation between the Riemann-Liouville fractional integral (1) and Caputo fractional derivative (4) to estimate the initial state.

$$\left(I^\alpha D^\alpha f\right)(t) = f(t) - \sum_{k=0}^{[\alpha]-1} \frac{f^{(k)}(0+)}{k!} t^k, \quad t \geq 0, n-1 < \alpha \leq n \quad (5)$$

Here $[\alpha]$ is the smallest integer larger than α .

2.2 Block impulse functions and Fractional Integral Operational Matrix

The set of BPFs defined in the semi-open interval $[0, T)$ is defined as [21].

$$b_m(t) = \begin{cases} 1, & (m-1)h \leq t < mh \\ 0, & \text{otherwise} \end{cases} \quad (6)$$

Here $h = T/M$, $m = 1, \dots, M$ and M are the number of BPFs in the set. Then $B_M(t) = [b_1(t), \dots, b_M(t)]^T$ is called a block impulse function.

Any absolutely integrable function $f(t)$ defined in interval $[0, T)$ is expressed as a linear combination of BPFs as follows:

$$f(t) = \sum_{m=1}^M \left[\frac{1}{h} \int_{(m-1)h}^{mh} f(t) dt \right] b_m(t) \approx \sum_{m=1}^M f_m b_m(t) = F^T B_N(t) \quad (7)$$

Here, $F = (f_1, \dots, f_M)^T$ and $f_m (m=1, \dots, M)$ are :

$$f_m = \frac{1}{h} \int_{(m-1)h}^{mh} f(t) dt \approx f((m-1)h), \quad m=1, \dots, M$$

Then the Riemann-Liouville fractional integral of BPFs can be approximated as [16] :

$$I^\alpha B_M(t) \approx F^\alpha B_M(t) \quad (8)$$

Here

$$F^\alpha = \frac{h^\alpha}{\Gamma(\alpha+2)} \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \dots & \xi_N \\ 0 & \xi_1 & \xi_2 & \dots & \xi_{N-1} \\ 0 & 0 & \xi_1 & \dots & \xi_{N-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \xi_1 \end{pmatrix} \quad (9)$$

$$h = T/M, \quad \xi_1 = 1, \xi_m = m^{\alpha+1} - 2(m-1)^{\alpha+1} + (m-2)^{\alpha+1}, m=2, \dots, M$$

The matrix defined by Eq. (8) is called fractional integral matrix (FIOM). Thus, the fractional integral of the function can be written using FIOM (7) as follows:

$$I^\alpha f(t) \approx F^T F^\alpha B_M(t) \quad (10)$$

In this way, the fractional integral operation is transformed into an algebraic operation.

3. Identification of Nonlinear Hammerstein-Wiener Fractional Order Systems with Nonzero Initial Conditions

Consider the SISO nonlinear Hammerstein-Wiener fractional system as follows.

$$D^{\alpha_N} f_N(x(t)) + \dots + a_1 D^{\alpha_1} f_1(x(t)) = b_K D^{\beta_K} g_K(u(t)) + \dots + b_1 D^{\beta_1} g_1(u(t))$$

$$y(t) = x(t) + v(t)$$

or

$$D^{\alpha_N} f_N(x(t)) + \sum_{n=1}^{N-1} a_n D^{\alpha_n} f_n(x(t)) = \sum_{k=1}^K b_k D^{\beta_k} g_k(u(t))$$

$$y(t) = x(t) + v(t) \quad (11)$$

Where $a_n, b_k (n=1, \dots, N, k=1, \dots, K)$ is the main parameters of the system to be estimated, $y(t)$ is the output signal, $v(t)$ is the measurement noise, $u(t)$ is the input signal, and $x(t)$ is the state of the system.

Without loss of generality, assume that

$$\alpha_N \geq \alpha_{N-1} \geq \dots \geq \alpha_1, \quad \beta_K \leq \beta_{K-1} \leq \dots \leq \beta_1, \quad \alpha_N > \beta_1$$

Also, if $L-1 < \alpha_N \leq L$ is assumed, the initial condition to be estimated is given by

$$y(0+) = y_0, \quad y'(0+) = y_{1,0}, \dots, y^{(L-1)}(0+) = y_{L-1,0} \quad (12)$$

First, we introduce the α_N order Riemann-Liouville fractional integral operation on both sides of the first expression of Eq. (11).

Then Eq. (11) becomes

$$I^{\alpha_N} D^{\alpha_N} f_N(x(t)) + \sum_{n=1}^{N-1} a_n I^{\alpha_N} D^{\alpha_n} f_n(x(t)) = \sum_{k=1}^K b_k I^{\alpha_N} D^{\beta_k} g_k(u(t)) \quad (13)$$

Then, by the property of Riemann-Liouville integral operator (2), Eq. (13) can be written as

$$I^{\alpha_N} D^{\alpha_N} f_N(x(t)) + \sum_{n=1}^{N-1} a_n I^{\alpha_N - \alpha_n} I^{\alpha_n} D^{\alpha_n} f_n(x(t)) = \sum_{k=1}^K b_k I^{\alpha_N - \beta_k} I^{\beta_k} D^{\beta_k} g_k(u(t)) \quad (14)$$

By the relation (5), the $I^{\alpha_n} D^{\alpha_n} f_n(x(t))$, $I^{\beta_k} D^{\beta_k} g_k(u(t))$, $n=1, \dots, N$, $k=1, \dots, K$ in the above equation can be written as

$$I^{\alpha_n} D^{\alpha_n} f_n(x(t)) = f_n(x(t)) - \sum_{i=0}^{[\alpha_n]-1} f_n^{(i)}(y(0+)) \frac{t^i}{i!}, \quad n=1, \dots, N$$

$$I^{\beta_k} D^{\beta_k} g_k(u(t)) = g_k(u(t)) - \sum_{j=1}^{[\beta_k]-1} g_k^{(j)}(u(0+)) \frac{t^j}{j!}, \quad k=1, \dots, K \quad (15)$$

Equation (15) can be written by Eq. (3) as follows :

$$\begin{aligned} I^{\alpha_n} D^{\alpha_n} f_n(x(t)) &= f_n(x(t)) - \sum_{i=0}^{[\alpha_n]-1} f_n^{(i)}(y(0+)) I^i(1), \quad n=1, \dots, N \\ I^{\beta_k} D^{\beta_k} g_k(u(t)) &= g_k(u(t)) - \sum_{j=1}^{[\beta_k]-1} g_k^{(j)}(u(0+)) I^j(1), \quad k=1, \dots, K \end{aligned} \quad (16)$$

Using Eq. (16), Eq. (14) becomes

$$\begin{aligned} f_N(x(t)) - \sum_{i=0}^{[\alpha_N]-1} f_N^{(i)}(y(0+)) I^i(1) + \sum_{n=1}^{N-1} a_n I^{\alpha_N - \alpha_n} \left[f_n(x(t)) - \sum_{i=0}^{[\alpha_n]-1} f_n^{(i)}(y(0+)) I^i(1) \right] \\ = \sum_{k=1}^K b_k I^{\alpha_N - \beta_k} \left[g_k(u(t)) - \sum_{j=1}^{[\beta_k]-1} g_k^{(j)}(u(0+)) I^j(1) \right] \end{aligned} \quad (17)$$

Let $x(t) \approx X^T B_M(t)$, $u(t) \approx U^T B_M(t)$ be in Eq. (17) by relation (7). Here, $X = (x_1, \dots, x_M)^T$, $U = (u_1, \dots, u_M)^T$ and $x_m = x(mh)$, $u_m = u(mh)$ are. Then Eq. (17) becomes

$$\begin{aligned} f_N(X^T B_M(t)) - \sum_{i=0}^{[\alpha_N]-1} f_N^{(i)}(y(0+)) I^i(I_1^T B_M(t)) \\ + \sum_{n=1}^{N-1} a_n I^{\alpha_N - \alpha_n} \left[f_n(X^T B_M(t)) - \sum_{i=0}^{[\alpha_n]-1} f_n^{(i)}(y(0+)) I^i(I_1^T B_M(t)) \right] \\ = \sum_{k=1}^K b_k I^{\alpha_N - \beta_k} \left[g_k(U^T B_M(t)) - \sum_{j=1}^{[\beta_k]-1} g_k^{(j)}(u(0+)) I^j(I_1^T B_M(t)) \right] \end{aligned} \quad (18)$$

Where $I_1 = (1, \dots, 1)^T \in R^M$.

In Eq. (18), $f_n(X^T B_M(t))$, $g_k(U^T B_M(t))$, $n=1, \dots, N$, $k=1, \dots, K$ is established by the following relation:

$$\begin{aligned} f_n(X^T B_M(t)) &= (f_n(x_1), \dots, f_n(x_M))^T B_M(t), \quad n=1, \dots, N \\ g_k(U^T B_M(t)) &= (g_k(u_1), \dots, g_k(u_M))^T B_M(t), \quad k=1, \dots, K \end{aligned} \quad (19)$$

Let us now construct the following expression in (19) :

$$\begin{aligned} (f_n(x_1), \dots, f_n(x_M))^T &= (x_{1(f_n)}, \dots, x_{M(f_n)})^T = X_{(f_n)}^T, \quad n=1, \dots, N \\ (g_k(u_1), \dots, g_k(u_M))^T &= (u_{1(g_k)}, \dots, u_{M(g_k)})^T = U_{(g_k)}^T, \quad k=1, \dots, K \end{aligned}$$

Then Eq. (18) becomes

$$\begin{aligned} X_{(f_N)}^T B_M(t) - \sum_{i=0}^{[\alpha_N]-1} f_N^{(i)}(y(0+)) I^i(I_1^T B_M(t)) \\ + \sum_{n=1}^{N-1} a_n I^{\alpha_N - \alpha_n} \left[X_{(f_n)}^T B_M(t) - \sum_{i=0}^{[\alpha_n]-1} f_n^{(i)}(y(0+)) I^i(I_1^T B_M(t)) \right] \\ = \sum_{k=1}^K b_k I^{\alpha_N - \beta_k} \left[U_{(g_k)}^T B_M(t) - \sum_{j=1}^{[\beta_k]-1} g_k^{(j)}(u(0+)) I^j(I_1^T B_M(t)) \right] \end{aligned} \quad (20)$$

Equation (20) can be written by the relation (10) as follows:

$$\begin{aligned} X_{(f_N)}^T B_M(t) - \sum_{i=0}^{[\alpha_N]-1} f_N^{(i)}(y(0+)) I_1^T F^i B_M(t) \\ + \sum_{n=1}^{N-1} a_n F^{\alpha_N - \alpha_n} \left[X_{(f_n)}^T B_M(t) - \sum_{i=0}^{[\alpha_n]-1} f_n^{(i)}(y(0+)) I_1^T F^i B_M(t) \right] \\ = \sum_{k=1}^K b_k F^{\alpha_N - \beta_k} \left[U_{(g_k)}^T B_M(t) - \sum_{j=1}^{[\beta_k]-1} g_k^{(j)}(u(0+)) I_1^T F^j B_M(t) \right] \end{aligned}$$

or

$$\begin{aligned}
 X_{(f_N)}^T B_M(t) - \sum_{i=0}^{[\alpha_N]-1} f_N^{(i)}(y(0+)) I_1^T F^i B_M(t) \\
 + \sum_{n=1}^{N-1} a_n X_{(f_n)}^T F^{\alpha_N - \alpha_n} B_M(t) - \sum_{n=1}^{N-1} \sum_{i=0}^{[\alpha_n]-1} a_n f_n^{(i)}(y(0+)) I_1^T F^{\alpha_N - \alpha_n + i} B_M(t) \\
 = \sum_{k=1}^K b_k U_{(g_k)}^T F^{\alpha_N - \beta_k} B_M(t) - \sum_{k=1}^{K-1} \sum_{j=0}^{[\beta_k]-1} b_k g_k^{(j)}(u(0+)) I_1^T F^{\alpha_N - \beta_k + j} B_M(t)
 \end{aligned} \quad (21)$$

Let us now construct the following notation in Eq. (21):

$$\begin{aligned}
 d_{in} &= \begin{cases} 1, & i < [\alpha_n] \\ 0, & i \geq [\alpha_n] \end{cases}, \quad c_{jk} = \begin{cases} 1, & j < [\beta_k] \\ 0, & j \geq [\beta_k] \end{cases} \\
 p_{in} &= d_{in} f_n^{(i)}(y(0+)), \quad q_{jk} = c_{jk} g_k^{(j)}(u(0+)), \quad r_i = f_N^{(i)}(y(0+))
 \end{aligned} \quad (22)$$

Then Eq. (21) becomes

$$\begin{aligned}
 X_{(f_N)}^T B_M(t) + \sum_{n=1}^{N-1} a_n X_{(f_n)}^T F^{\alpha_N - \alpha_n} B_M(t) - \sum_{i=0}^{[\alpha_N]-1} r_i I_1^T F^i B_M(t) - \sum_{n=1}^{N-1} \sum_{i=0}^{[\alpha_n]-1} p_{in} I_1^T F^{\alpha_N - \alpha_n + i} B_M(t) \\
 = \sum_{k=1}^K b_k U_{(g_k)}^T F^{\alpha_N - \beta_k} B_M(t) - \sum_{k=1}^{K-1} \sum_{j=0}^{[\beta_k]-1} q_{jk} I_1^T F^{\alpha_N - \beta_k + j} B_M(t)
 \end{aligned}$$

or

$$\begin{aligned}
 \left(X_{(f_N)}^T + \sum_{n=1}^{N-1} a_n X_{(f_n)}^T F^{\alpha_N - \alpha_n} - \sum_{i=0}^{[\alpha_N]-1} r_i I_1^T F^i - \sum_{n=1}^{N-1} \sum_{i=0}^{[\alpha_n]-1} p_{in} I_1^T F^{\alpha_N - \alpha_n + i} \right) B_M(t) \\
 = \left(\sum_{k=1}^K b_k U_{(g_k)}^T F^{\alpha_N - \beta_k} - \sum_{k=1}^{K-1} \sum_{j=0}^{[\beta_k]-1} q_{jk} I_1^T F^{\alpha_N - \beta_k + j} \right) B_M(t)
 \end{aligned} \quad (23)$$

The parameters to be estimated in Eq. (23) are:

$$a_n, b_k, r_i, p_{in}, q_{jk}, (n=1, \dots, N, k=1, \dots, K, i=1, \dots, [\alpha_N]-1, j=1, \dots, [\beta_K]-1) \quad (24)$$

Here, $a_n, b_k, (n=1, \dots, N, k=1, \dots, K)$ are the main parameters and the initial states (12) are obtained by $p_{in}, q_{jk}, (n=1, \dots, N, k=1, \dots, K, i=1, \dots, [\alpha_N]-1, j=1, \dots, [\beta_K]-1)$.

When we know all fractional orders by Eq. (23), we can write the linear regression equation to estimate the parameter (24) as follows:

$$\begin{aligned}
 X_{(f_N)}(z) = - \sum_{n=1}^{N-1} a_n X_{(f_n)}^T F^{\alpha_N - \alpha_n}(z) + \sum_{i=0}^{[\alpha_N]-1} r_i I_1^T F^i(z) + \sum_{n=1}^{N-1} \sum_{i=0}^{[\alpha_n]-1} p_{in} I_1^T F^{\alpha_N - \alpha_n + i}(z) \\
 + \sum_{k=1}^K b_k U_{(g_k)}^T F^{\alpha_N - \beta_k}(z) - \sum_{k=1}^{K-1} \sum_{j=0}^{[\beta_k]-1} q_{jk} I_1^T F^{\alpha_N - \beta_k + j}(z), \quad z=1, \dots, M
 \end{aligned} \quad (25)$$

Here $X_{(f_N)}(z)$ is the z -th element of the vector $X_{(f_N)}$ and $F^{\alpha_N - \alpha_n}(z), F^i(z), F^{\alpha_N - \alpha_n + i}(z), F^{\alpha_N - \beta_k}(z), F^{\alpha_N - \beta_k + j}(z)$ are the z -th column vectors of the corresponding matrix, respectively.

Since the output signal $y(t)$ is used for parameter estimation, the regression model (25) can be written by the second expression of Eq. (11) as follows:

$$\begin{aligned}
 Y_{(f_N)}(z) = - \sum_{n=1}^N a_n Y_{(f_n)}^T F^{\alpha_N - \alpha_n}(z) + \sum_{k=1}^K b_k U_{(g_k)}^T F^{\alpha_N - \beta_k}(z) + \sum_{i=0}^{[\alpha_N]-1} r_i I_1^T F^i(z) \\
 + \sum_{n=1}^{N-1} \sum_{i=0}^{[\alpha_n]-1} p_{in} I_1^T F^{\alpha_N - \alpha_n + i}(z) - \sum_{k=1}^{K-1} \sum_{j=0}^{[\beta_k]-1} q_{jk} I_1^T F^{\alpha_N - \beta_k + j}(z) + W(z), \quad z=1, \dots, M
 \end{aligned} \quad (26)$$

Here, $W(z)$ is obtained by the measurement noise $v(t) = V^T B_M(t)$ as follows:

$$W(z) = V_{(f_N)}(z) + \sum_{n=1}^{N-1} a_n V_{(f_n)}^T F^{\alpha_N - \alpha_n}(z), \quad z=1, \dots, M \quad (27)$$

For simplicity, we introduce the following notation.

$$\begin{aligned}\theta_n^1 &= a_n, \quad Y_n^1(z) = -Y_{(f_n)}^T F^{\alpha_N - \alpha_n}(z), \quad (n=1, \dots, N-1) \\ \theta_k^2 &= b_k, \quad Y_k^2(z) = U_{(g_k)}^T F^{\alpha_N - \beta_k}(z), \quad (k=1, \dots, K) \\ \theta_i^3 &= r_i, \quad Y_i^3(z) = I_1^T F^i(z), \quad (i=1, \dots, [\alpha_N] - 1) \\ \theta_{in}^4 &= p_{in}, \quad Y_{in}^4(z) = I_1^T F^{\alpha_N - \alpha_n + i}(z), \quad (i=1, \dots, [\alpha_N] - 1, n=1, \dots, N-1) \\ \theta_{jk}^5 &= q_{jk}, \quad Y_{jk}^5(z) = -I_1^T F^{\alpha_N - \beta_k + j}(z), \quad (k=1, \dots, K-1, j=1, \dots, [\beta_K] - 1)\end{aligned}$$

Then Eq. (26) becomes

$$Y_{(f_N)}(z) = \sum_{n=1}^N \theta_n^1 Y_n^1(z) + \sum_{k=1}^K \theta_k^2 Y_k^2(z) + \sum_{i=0}^{[\alpha_N]-1} \theta_i^3 Y_i^3(z) + \sum_{n=1}^{N-1} \sum_{i=0}^{[\alpha_N]-1} \theta_{in}^4 Y_{in}^4(z) + \sum_{k=1}^{K-1} \sum_{j=0}^{[\beta_K]-1} \theta_{jk}^5 Y_{jk}^5(z) + W(z), \quad (z=1, \dots, M) \quad (28)$$

Then, the parameters to be estimated are :

$$\theta = (\theta_1^1, \dots, \theta_N^1, \theta_1^2, \dots, \theta_K^2, \theta_0^3, \dots, \theta_{[\alpha_N]-1}^3, \theta_{01}^4, \dots, \theta_{(N-1)([\alpha_N]-1)}^4, \theta_{01}^5, \dots, \theta_{(K-1)([\beta_K]-1)}^5)^T \quad (29)$$

Then the $z(=1, \dots, M)$ -th column vector $\varphi(z)$ of the data matrix φ is given by

$$\begin{aligned}\varphi(z) &= (Y_1^1(z), \dots, Y_N^1(z), Y_1^2(z), \dots, Y_K^2(z), Y_0^3(z), \dots, Y_{[\alpha_N]-1}^3(z), Y_{01}^4(z), \dots, Y_{[\alpha_N]-1, N-1}^4(z), \\ &\quad Y_{01}^5(z), \dots, Y_{(K-1)([\beta_K]-1)}^5(z))^T, \quad (z=1, \dots, M)\end{aligned}$$

Thus, Eq. (28) can be expressed as

$$Y_{(f_N)}(z) = \theta^T \varphi(z)$$

If the fractional orders of the system are unknown, then the optimal estimates of the system parameters can be obtained by minimizing the following objective function:

$$\begin{aligned}& (\hat{a}_i^*, \hat{b}_j^*, \hat{\alpha}_i^*, \hat{\beta}_j^*, \hat{x}^*(0), \dots, \hat{x}^{([\alpha_N]-1)^*}(0)) \\ &= \arg \min_{\hat{a}_i^*, \hat{b}_j^*, \hat{\alpha}_i^*, \hat{\beta}_j^*, \hat{x}^*(0), \dots, \hat{x}^{([\alpha_N]-1)^*}(0) \in \Omega} \frac{1}{2} \sum_{z=1}^M (Y_{(f_N)}(z) - \hat{\theta}^T \varphi(\hat{a}_i^*, \hat{\beta}_j^*, z))^2\end{aligned} \quad (30)$$

Here Ω is the feasible search region for the system parameters. To solve this problem, we use the fmincon function with the interior point method, a standard optimization tool of MATLAB. This method can solve the large-scale optimization problem and is more efficient and faster than the SQP method. If the fractional orders $\alpha_i, \beta_j, (i=1, \dots, N, j=1, \dots, K)$ of the system are known, we can use the following bias compensated recursive least squares (BCRLS) algorithm without estimator bias to estimate the parameter of the linear regression model (28) [30].

$$\begin{aligned}L(z) &= P(z-1)\varphi(z)[\lambda + \varphi^T(z)P(z-1)\varphi(z)]^{-1} \\ P(0) &= p_0 I \\ \varepsilon(z) &= Y_{(f_N)}(z) - \varphi^T(z)\hat{\theta}_{RLS}(z-1) \\ \hat{\theta}_{RLS}(z) &= \hat{\theta}_{RLS}(z-1) + L(z)\varepsilon(z), \quad \hat{\theta}_{RLS}(0) = \frac{1}{p_0} I \\ P(z) &= [I - L(z)\varphi^T(z)]^{-1} P(z-1) / \lambda \\ \theta_{BCRLS}(z) &= \theta_{RLS}(z-1) - P(z) \left[\sum_{l=1}^z \varphi(l)(Y_{(f_N)}(l) - \varphi^T(l)\hat{\theta}_{RLS}(l)) \right]\end{aligned} \quad (31)$$

Here $P(0) = \mathcal{I}$, ($\gamma = 10^3 \sim 10^8$), I is the identity matrix of the corresponding dimension and $0.2 < \lambda < 1$.

And $\hat{\theta}_{RLS}(0) = \frac{1}{\gamma} I$.

4. Simulation examples

In this section, we demonstrate the effectiveness of the proposed method with two examples. To evaluate the identification accuracy, we use the following error norm:

$$e(t) = \|\theta - \hat{\theta}(t)\|_2 \quad (32)$$

Here θ is the real parameter vector and $\hat{\theta}$ is the estimated parameter vector.

Matlab 2023b was used for the simulation.

Example 1

Consider the following example.

$$\begin{aligned} (D^{1.5}x^2)(t) + 0.5(D^{1.3}x)(t) + 0.2x(t) &= 0.7(D^{0.6}u^2)(t) + 2(D^{0.1}u)(t) \\ y(t) &= x(t) + v(t) \end{aligned} \quad (33)$$

The initial conditions are:

$$x(0) = 1.5, \quad x'(0) = 1 \quad (34)$$

The main parameters to be estimated in nonlinear fractional systems (33) are

$$a_2 = 0.5, \quad a_1 = 0.2, \quad b_1 = 0.7, \quad b_2 = 2 \quad (35)$$

and initial conditions (34).

For identification, the following input signal is used:

$$u(t) = 0.7 \sin 0.12t + 0.3 \cos(0.2t - \pi/3) - 1.2 \sin 0.5t$$

Figure 1 shows the input signal and Figure 2 shows the output signal for the nonlinear fractional order system (33) in the case of the output SNR = 20 dB Gaussian white noise.

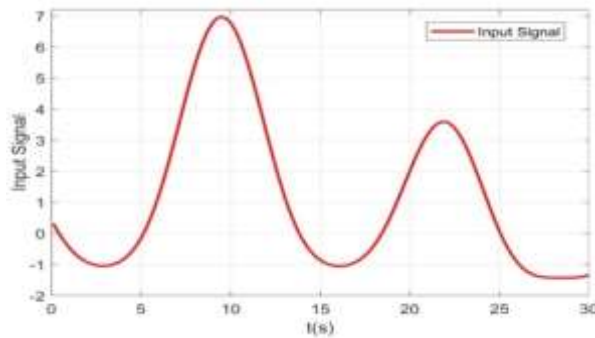


Figure 1. Input Signal of Example 1

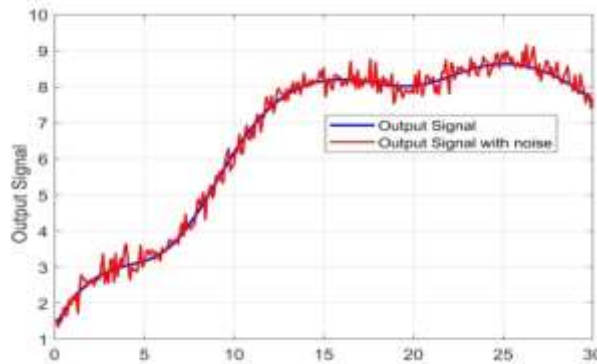


Figure 2. Output Signal of Example 1

Then, by Eq. (17), Eq. (20) becomes

$$\begin{aligned} [x^3(t) - x_0^3 - 2x(0)x'(0)I^1(1)] + a_1 I^{0.2}[x(t) - x(0) - x'(0)I^1(1)] + 0.2 I^{1.5}x(t) \\ = b_1(I^{0.9}u^3)(t) + b_2(I^{1.4}u)(t) \end{aligned} \quad (36)$$

Since $f_3(x(t)) = x^2(t)$, $f_2(x(t)) = x(t)$, $f_1(x(t)) = x(t)$, $g_1(u(t)) = u^2(t)$, $g_2(u(t)) = u(t)$ in Eq. (33), it is expressed as

$$\begin{aligned} f_3(x(t)) &= (x_1^2 \quad \cdots \quad x_M^2)B(t) = X_{(x^2)}^T B(t) \\ f_2(x(t)) &= f_1(x(t)) = (x_1 \quad \cdots \quad x_M)B(t) = X^T B(t) \\ g_1(u(t)) &= (u_1^2 \quad \cdots \quad u_M^2)B(t) = U_{(u^2)}^T B(t) \\ g_2(u(t)) &= (u_1 \quad \cdots \quad u_M)B(t) = U^T B(t) \end{aligned}$$

when $x(t) \approx (x_1 \quad \cdots \quad x_M)B(t) = X^T B(t)$, $u(t) \approx (u_1 \quad \cdots \quad u_M)B(t) = U^T B(t)$ is used.

Thus, Eq. (36) can be written by (25) as follows:

$$\begin{aligned} X_{(x^2)}^T &= -a_1 X^T F^{0.2} - a_2 X^T F^{1.5} + b_1 U_{(u^2)}^T F^{0.9} + b_2 U^T F^{1.4} \\ &\quad + x^2(0) + 2x(0)x'(0)I_1^T F^1 + a_1 x(0)I_1^T F^{0.2} + a_1 x'(0)I_1^T F^{1.2} \end{aligned}$$

Where $I_1 = (1 \quad \cdots \quad 1)^T \in R^M$.

In the above equation, if $c_1 = x^2(0)$, $c_2 = 2x(0)x'(0)$, $c_3 = a_1 x(0)$, $c_4 = a_1 x'(0)$, the parameter vector is

$$\theta = (a_1, a_2, b_1, b_2, c_1, c_2, c_3, c_4)^T$$

Then, the initial condition is obtained by

$$x(0) = c_3 / a_1, x'(0) = c_4 / a_1 \quad (37)$$

Then, according to the BCRLS algorithm (31), a_1, a_2, b_1, b_2 are estimated online and initial conditions are obtained by Eq. (37). Figure 3 shows the parameter estimation curve.

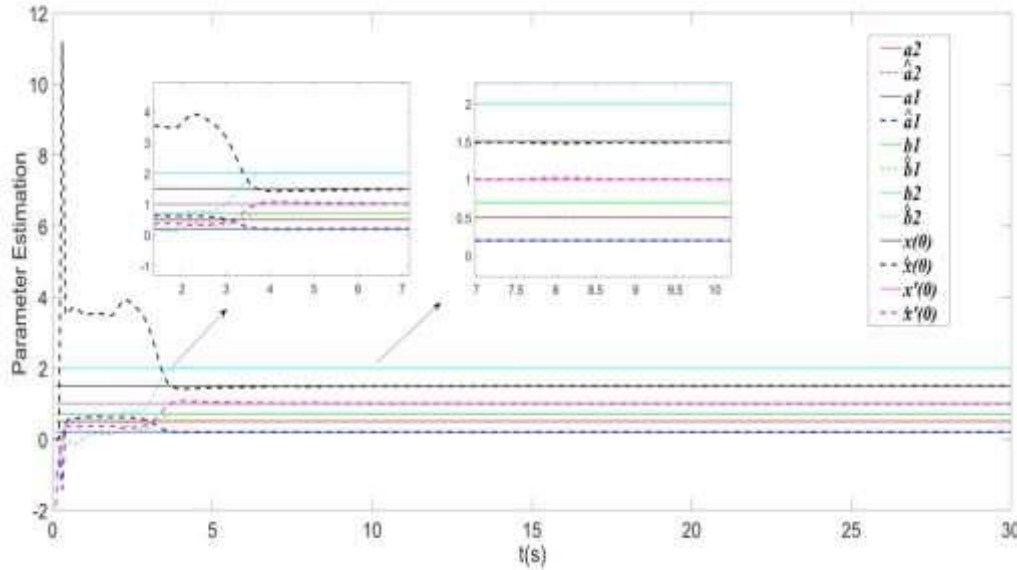


Figure 3. Parameter Estimation Curve of Example 1

As shown in Fig. 3, it is shown that the proposed method estimates the actual parameters accurately. Figure 4 shows the curves according to the error criterion (32).

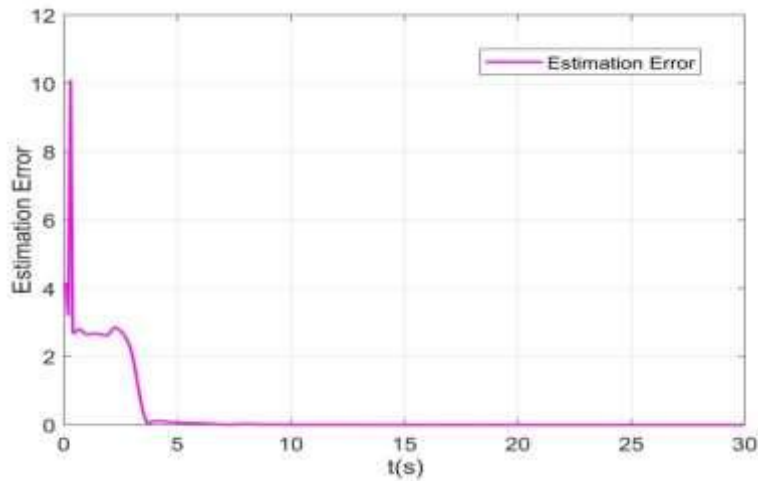


Figure 4. Parameter Estimation Error of Example 1

The output prediction error for the estimated parameters is shown in Fig. 5.

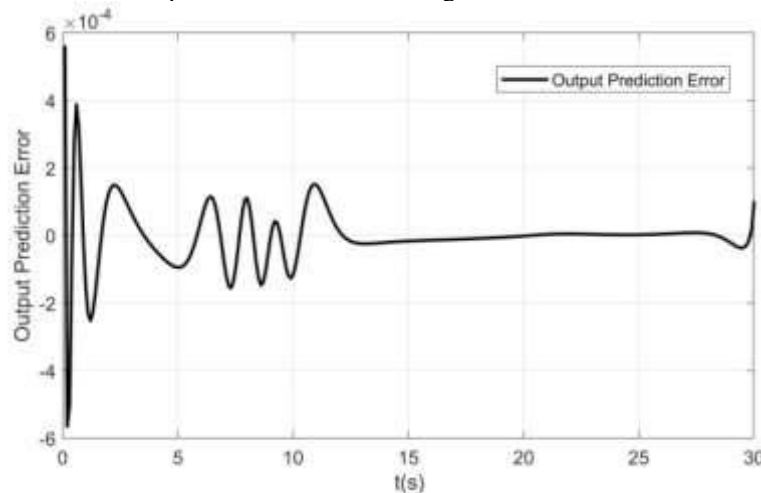


Figure 5. Output Prediction Error of Example 1

Table 1 shows the parameter estimation characteristics for $M = 300, 1000$, $SNR = 20, 30dB$. Parameter estimates were used to average the values obtained at $t = T$ by performing $N = 500$ times simulations.

Table 1. Parameter estimation characteristics for $M = 300, 1000$, $SNR = 20, 30dB$ (if fractional order is unknown)

	True value	$M = 300$		$M = 1000$	
		$SNR = 20dB$	$SNR = 30dB$	$SNR = 20dB$	$SNR = 30dB$
a_2	0.5	0.4871	0.4997	0.4993	0.4999
a_1	0.2	0.2013	0.2003	0.2009	0.2001
b_1	0.7	0.6891	0.6986	0.6999	0.6999
b_2	2	2.0013	2.0009	2.0010	2.0007
$x(0)$	1.5	1.4901	1.5018	1.5001	1.5000
$x'(0)$	1	0.9901	0.9987	0.9965	0.9999

As shown in Table 1, it can be seen that the estimation accuracy increases with increasing number of BPFs number M , and the estimation accuracy is guaranteed at various noise levels.

Table 2 shows the estimation performance when the fractional order of the system is unknown. The parameter estimates were used to average values using $N = 50$ time simulations.

Table 2. Parameter estimation characteristics for $M = 200, 300$, $SNR = 20, 30dB$ (if fractional order is unknown)

	True value	$M = 200$		$M = 300$	
		$SNR = 20dB$	$SNR = 30dB$	$SNR = 20dB$	$SNR = 30dB$
α_3	1.5	1.4355	1.4961	1.4732	1.5024
α_2	1.3	1.3412	1.3012	1.2931	1.2991
α_1	0	0.0021	0.0014	0.0019	0.0011
β_1	0.6	0.5783	0.5892	0.5917	0.6002
β_2	0.1	0.0902	0.0934	0.0921	0.0998
a_2	0.5	0.4755	0.4741	0.4891	0.4898
a_1	0.2	0.1902	0.2014	0.1983	0.2011
b_1	0.7	0.6281	0.6781	0.6901	0.7103
b_2	2	2.0142	2.0103	1.9802	2.0007
$x(0)$	1.5	1.4781	1.4942	1.4981	1.5001
$x'(0)$	1	0.9671	0.9822	0.9855	0.9987

Table 2 shows that the parameters are estimated efficiently for the objective function (30) for optimization. In the case of the system fractional order unknown, the parameter estimation accuracy is less than in the case of the fractional order known.

Fig. 6 shows the output signal under nonzero initial and zero initial conditions in the absence of output noise.

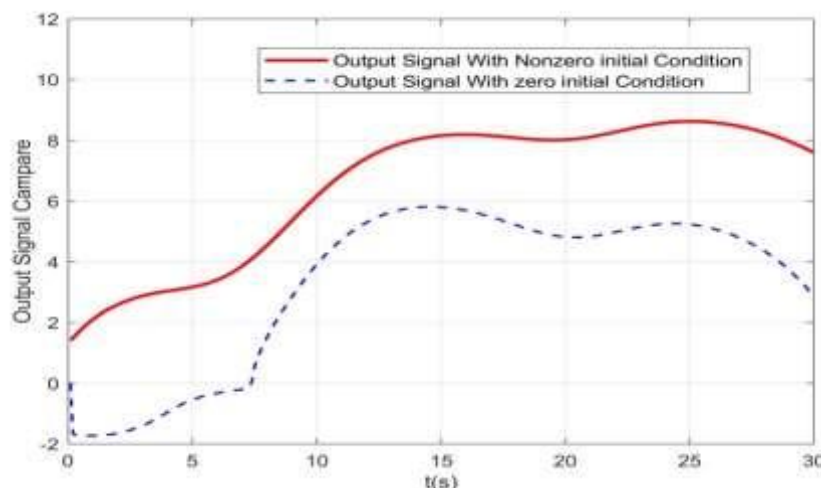


Figure 6 Output signal of Example 1 under Nonzero Initial and

Zero Initial Conditions in the Absence of Noise

As can be seen in Fig. 5, the output characteristics differ greatly with respect to the initial conditions. Thus, in identifying nonlinear Hammerstein-Wiener fractional systems with nonzero initial conditions, the nonzero initial condition must be considered.

Example 2

Next, consider the identification problem for a simple pendulum shown in [28].

$$\begin{cases} (D^{0.95}x_1)(t) = x_2(t) \\ (D^{0.95}x_2)(t) = -\frac{g}{l}\sin x_1(t) + u(t) \\ y(t) = x_1(t) + v(t) \end{cases} \quad (38)$$

where $x_1 = \theta$ is the angular displacement and $x_2 = \omega$ is the angular velocity. $g = 9.81 \text{ m/s}^2$ is the gravitational acceleration, $l = 0.5 \text{ m}$ is the length of the pendulum.

The initial state is $x_1(0) = \pi/2$, $x_2(0) = 0$.

The simulation time gives $[0, 20)$ and the number of BPFs gives $M = 200$.

This system assumes no air resistance and no contact friction of the pendulum. The input signal is $u(t) = 15\eta(t)$. Here, $\eta(t)$ is a unit step function.

For identification, applying Riemann-Liouville's 0.59 fractional integral on both sides of Eq. (38) yields

$$\begin{cases} x_1(t) - x_1(0) = (I^{0.95}x_2)(t) \\ x_2(t) - x_2(0) = -\frac{g}{l}(I^{0.95}\sin x_1)(t) + (I^{0.95}u)(t) \end{cases} \quad (39)$$

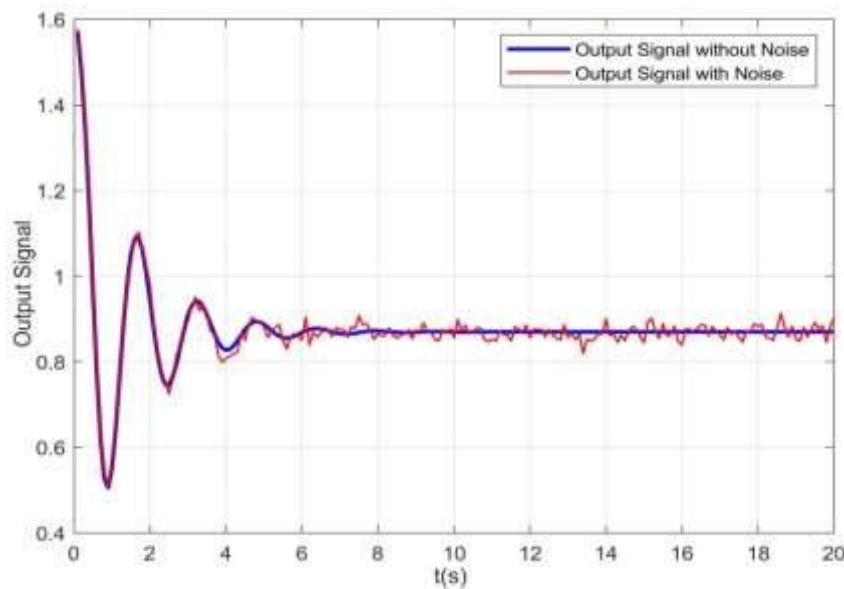


Figure 7. Output Signal of Example

Substituting $x_2(t)$ in the second equation of Eq. (39) into the first equation yields:

$$x_1(t) = -\frac{g}{l}(I^{1.9}\sin x_1)(t) + x_1(0) + I^{0.95}(x_2(0)) + (I^{1.9}u)(t)$$

Since $x_2(0) = x_1'(0)$, the above equation becomes

$$x_1(t) = -\frac{g}{l}(I^{1.8}\sin x_1)(t) + x_1(0) + I^{0.95}(x_1'(0)) + (I^{1.9}u)(t) \quad (40)$$

The output signal is shown in Fig. 6 for the measurement noise-free and SNR = 20 dB measurement noise. Assuming

$$\begin{aligned} u(t) &= 15\eta(t) = 15(1 \ \cdots \ 1)^T B_M(t) \\ x_1(t) &\approx (x_1^1 \ \cdots \ x_1^M) B(t) = X^T B_M(t) \\ f_1(x_1(t)) &= \sin x_1(t) \approx (\sin(x_1^1) \ \cdots \ \sin(x_1^M)) B(t) = X_{(\sin x_1)}^T B_M(t) \end{aligned}$$

and $\theta_1 = -g/l$, $\theta_2 = x_1(0)$, $\theta_3 = x_1'(0)$ in Eq. (40), we have

$$X^T B_M(t) = \theta_1 X_{(\sin x_1)}^T I^{1.9} B_M(t) + \theta_2 I_1^T B_M(t) + \theta_3 I_1^T I^{0.95} B_M(t) + 15 I_1^T I^{1.9} B_M(t)$$

Where $I_1 = (1 \ \cdots \ 1)^T \in R^M$. Applying FIOM to the above equation, we have

$$X^T B_M(t) = [\theta_1 X_{(\sin x_i)}^T F^{1.9} + \theta_2 I_1^T + \theta_3 I_1^T F^{0.95} + 15 I_1^T F^{1.9}] B_M(t) \quad (41)$$

It can be seen that the initial condition AAA in Eq. (41) is directly estimated through the parameter θ_1, θ_2 . Fig. 8 shows the parameter estimation curve and Fig. 9 shows the error curve calculated by the error norm (32).

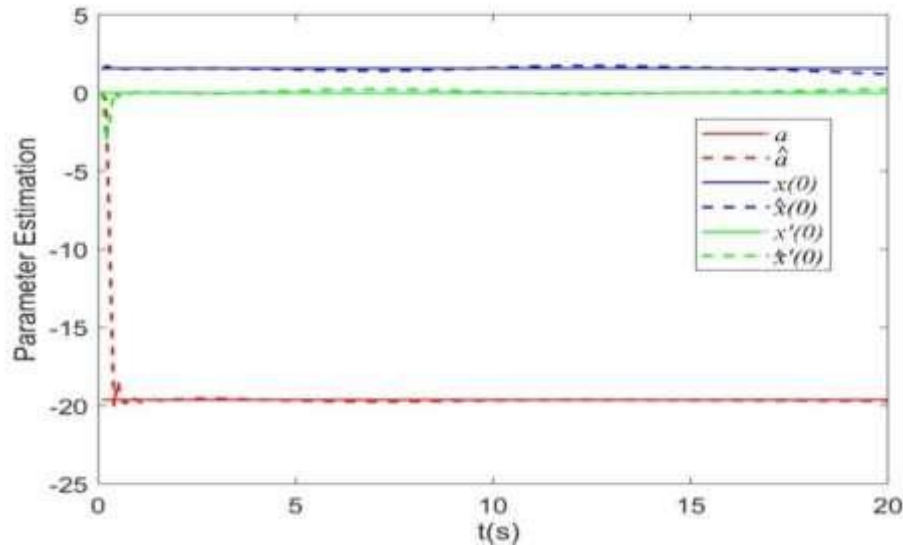


Figure8. Parameter Estimation Curve of Example 2

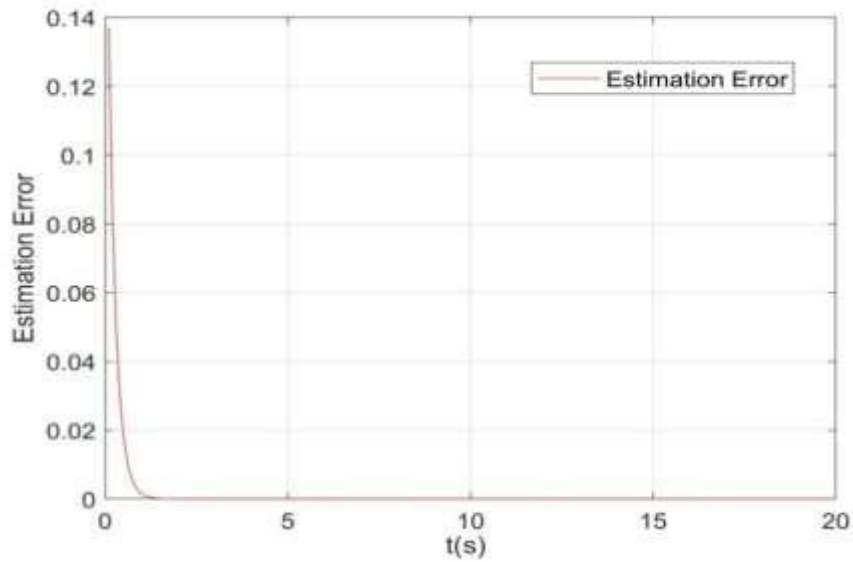


Figure 9. Parameter Estimation Error of Example 2

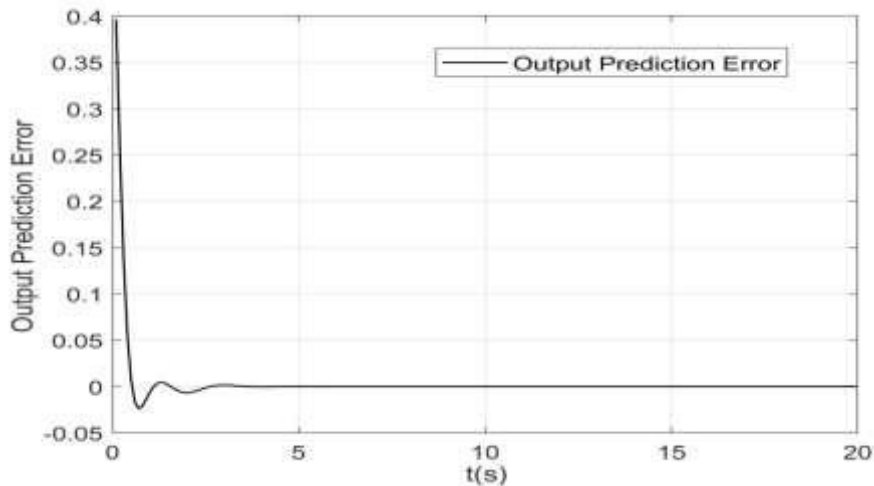


Figure 10. Output Prediction Error of Example 2

Fig. 10 shows the output prediction error for Example 2. As can be seen in Figs. 8 and 9, we can see that the estimation performance is better than that in [28]. It is also found that the method proposed in this paper is relatively much simpler than in [28]. Table 3 shows the parameter estimation characteristics for $M = 300, 1000$, $SNR = 20, 30dB$.

Parameter estimates were used to average the values obtained at $t = T$ by performing $N = 500$ time simulations.

Table 3. Parameter Estimation Characteristics for $M = 300, 1000$, $SNR = 20, 30dB$

	True value	$M = 300$		$M = 1000$	
		$SNR = 20dB$	$SNR = 30dB$	$SNR = 20dB$	$SNR = 30dB$
a	-19.62	-19.5931	-19.6151	-19.6201	-19.6199
$x(0)$	1.5708	1.5603	1.5689	1.5704	1.5693
$x'(0)$	0	0.0129	0.0022	0.0118	0.0017

As shown in Table 3, it can be seen that the estimation accuracy increases with increasing number of BPFs and the estimation accuracy is guaranteed at various noise levels. As in Example 1, we can see that the estimation accuracy increases with increasing number of BPFs M , and the estimation accuracy is guaranteed at various noise levels.

Table 4 shows the estimation performance when the fractional order of the system is unknown. The parameter estimates were used to average values using $N = 50$ time simulations.

Table4. Parameter estimation characteristics for $M = 200, 300$, $SNR = 20, 30dB$ (if fractional order is unknown)

	True value	$M = 200$		$M = 300$	
		$SNR = 20dB$	$SNR = 30dB$	$SNR = 20dB$	$SNR = 30dB$
α	0.95	0.9102	0.9422	0.9391	0.9491
a	-19.62	-18.98	-19.33	-19.31	-19.47
$x(0)$	1.5708	1.4981	1.5621	1.5699	1.5702
$x'(0)$	0	0.0213	0.0113	0.0106	0.0101

In Table 4, we show that the parameters are estimated efficiently for the objective function (30) for optimization.

5. Conclusion

A parameter identification method of a nonlinear Hammerstein-Wiener fractional order system with nonzero initial conditions is proposed. The simulation showed that the initial conditions have a significant effect on the output of the system. Simulation results show that the proposed method guarantees high estimation accuracy for nonzero initial conditions. The proposed method can be applied to the identification of many nonlinear fractional systems with nonzero initial conditions and can also be used for the identification of nonlinear fractional systems with multiple inputs. Although we have not considered the problem of improving identification accuracy by hybridization with various special polynomials, this problem is not essential. Also, there are few cases with input delay in real environment. Thus, the following goal is to develop an efficient identification algorithm that takes into account initial conditions and input delay.

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